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# Riemann–Cartan–Weyl Quantum Geometry. II Cartan Stochastic Copying Method, Fokker–Planck Operator and Maxwell–De Rham Equations

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We reintroduce the Riemann–Cartan–Weyl (RCW) spacetime geometries of quantum mechanics [Rapoport (1996), *Int. J. Theor. Phys.* **35**(2)] in two novel ways: first, through the covariant formulation of the Fokker–Planck operator of the quantum motions defined by these geometries, and second, by stochastic extension of Cartan's development method. The latter is a gauge-theoretic formulation of nonlinear diffusions in spacetime in terms of the stochastic differential geometry associated to the RCW geometries with Weyl torsion. The Weyl torsion plays the fundamental role of describing the first moment (incorporating also the fluctuations due to the second moment) of the stochastic diffusion processes. In this article we present the most general expression of the Weyl torsion one-form given in terms of its de Rham decomposition into the exact component associated with the 0-spin field $\psi$ and two electromagnetic potentials, one the codifferential of a 2-form and the other a harmonic 1-form. We thus give an original description of the Maxwell theory and its relation to torsion. We associate these electromagnetic potentials with the irreversibility of the diffusions. In an Appendix, we give a self-contained presentation of the theory of diffusions on manifolds and the stochastic calculi as a basis for the Cartan stochastic copying method.

## INTRODUCTION

In a previous article (Rapoport, 1996a) we introduced the Riemann–Cartan–Weyl (RCW) spacetime geometry, a metric $g$-compatible connection with torsion restricted to its trace component, associated to a trivial Weyl one-form of the form $Q = d \ln \psi$, where the scalar field $\psi$ introduces a

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conformal structure on the tangent space to the spacetime-manifold $M$; we further defined a family of Laplacian operators associated to these geometrical structures. These Laplacians were defined to act on differential forms of arbitrary degree, following Witten's conception of supersymmetric systems, in which forms of odd (even) degree represent fermions (bosons). These Laplacians were taken to define quantum fluctuational processes, since they define the infinitesimal generators of Markovian diffusion semigroups. We found that the quantum fluctuations described by higher than 0-spin fields depend solely in the data of the RCW connection: in the case of exact $Q$, these are $g$ and $\psi$.

These constructions can be extended to include nontrivial trace-torsion Weyl one-forms, i.e., $Q$ has, in addition to the exact one-form described above, nonexact terms. This extension is still rooted in scale transformations: In our introduction of the RCW geometries in Section 2 of Rapoport (1996a), we saw that one could start with a nonexact trace-torsion one-form, and through the lambda scale transformations due to Einstein and Kauffmann described there, one can obtain the full trace torsion one-form.

The first objective of this article is precisely the extension of our previous constructions to the case of the full trace-torsion one-form, which we shall determine by the procedure of applying a strong theorem of global analysis: the de Rham–Kodaira–Hodge decomposition of one-forms, with the additional requirement that the covariant backward Fokker–Planck operator has a time- (evolution parameter-) invariant probability density i.e., a density independent of the time-evolution parameter $\tau$ (which is not to be confused with the relativistic time coordinate). This approach will mark the appearance of two remarkable electromagnetic potentials, the codifferential of a 2-form and a harmonic 1-form, so both are divergenceless, and the harmonic form has a zero field (away from the nodes of $\psi$), and thus we shall name it the Aharonov–Bohm potential. This divergenceless character is precisely what defines in the stationary state a *conserved probability vector field* which characterizes the irreversibility of the fluctuations. Thus we shall have a gauge-theoretic characterization of irreversibility, i.e., the surgence of a *local* arrow of the time-evolution parameter related to the nonexact components of the trace-torsion. This gauge-theoretic characterization of the microscopic irreversibility of quantum fluctuations is new and contrasts with the approach of the Brussels–Austin groups, which relates the surgence of the time arrow to the existance of two analytical semigroups that appear in the study of large Poincaré systems. In the approach by Prigogine (1995) it is the presence of singularities in the continuous spectrum of classical and quantum systems which produces the time arrow; for a critical discussion of Prigogine's approach to irreversibility see Bricmont (1996).

Our presentation of this theory shall diverge from our previous article, in the sense that rather than positing the solution of taking the Laplacian of the RCW structure as the infinitesimal generator of the Markovian diffusion processes, we shall instead start from the opposite direction: Starting from the form of the infinitesimal generator of a probability-conserving Markovian semigroup, we shall derive the RCW geometries and their Laplacians from the mandatory requirement that the theory be formulated in intrinsic covariant form.

This has profound implications for our understanding of the quantum fluctuations, and we shall repeatedly discuss this point. Indeed, it is strongly believed that a possible conception of quantum fluctuations is to view them as classical deterministic motion (the drift vector fields) which are further *perturbed* by quantum noise described as Brownian processes. One uses precisely the fact that the drift vector field of the quantum fluctuations appears to be described in this theory incorporated into a *single* geometrical Laplacian as the conjugate (with respect to the metric $g$) vector field to the trace-torsion one-form, unifying thus the diffusion tensor (describing the metric) with the "classical" trace-torsion one-form $Q$, to give the average velocity of the quantum diffusion. Consequently, the Liouville operator of classical mechanics *cannot* be obtained from the stochastic theory by setting to zero the fluctuation terms of the Fokker–Planck operator, which in a noncovariant setting is, of course, valid (Gardiner, 1993). Thus, the ergodicity of these quantum fluctuational processes cannot be approached from the properties of the spectra of the Liouville operator since it cannot be intrinsically derived from the Fokker–Planck operator. Yet, most remarkably, the ergodicity structures of the classical theory, *can* be defined in the stochastic setting by noting a most remarkable property. The stochastic flows defined by integrating the quantum fluctuational motions are *diffeomorphisms* of spacetime! Consequently the usual ergodic approach through the stochastic extensions of the Perron–Frobenius semigroups and Lyapunov exponents can be developed naturally for the quantum fluctuations (Rapoport 1996c, d); this approach is more general than the one that stems from the Liouville or—more generally—the Perron–Frobenius operator of a classical dynamical system (Prigogine, 1995; Lasota and Mackey, 1985). It is to be remarked here that the ergodicity behavior of the quantum fluctuations in our geometrical setting will be related to topological properties of spacetime which are represented by a nonzero Aharonov–Bohm potential term in the trace-torsion (Rapoport, 1996d)

With regard to the geometrical status of the present formulation of quantum fluctuations, it is conceptually different from Nelson's stochastic mechanics, where the role of geometrical structures is subsidiary (Nelson, 1985; Namsrai, 1985; Guerra, 1981, de Witt-Morette and Elworthy, 1981). In Nelson's conception [and also in Bohm (1952) and Bohm and Vigier

(1953)], the drift is seen as a classical field independent of the fluctuations and no geometrical basis for the master equation (i.e., the Fokker–Planck equation) of the quantum diffusion is made explicit. Thus, in this geometrical sense, stochastic mechanics follows a trend common to the noncovariant formulations of diffusions and irreversible statistical mechanics. Let us quote from a standard treatise by Gardiner (1993, p. 235) "this independent description of fluctuations and deterministic motion is an embarrassment, and fluctuation-dissipation arguments are necessary to obtain some information about the fluctuations. In this respect, the master equation is a more complete description." Thus, in this article we follow to its roots the conception that it is the master operator which carries the complete information on the fluctuations, in keeping with the basic condition that the theory should be formulated intrinsically.[2] Then it is no surprise, with regard to Gardiner's comments above, that the present theory applies not only to quantum fluctuations, but also to nonlinear nonequilibrium systems: It is sufficient to regard the configuration manifold $M$ either as spacetime or as the differentiable manifold of the macroscopic variables of a generally nonlinear nonequilibrium thermodynamic system whose master operator is (one-half) the RCW Laplacian (Rapoport, 1996c, d). In this extended setting, the fluctuation-dissipation relations and a nonlinear Boltzmann theorem receive a very simple treatment in terms of the trace-torsion drift vector field (Rapoport, 1996d). Thus, the unified approach to the solution of the "embarrassment" pointed out by Gardiner appears to be universal.

Thus, it is relevant to discuss a method (like the second approach previously alluded to) in terms of which the introduction of the RCW geometry and its Laplacian appear as the basic structures for the characterization of fluctuations with continuous sample paths; this is the Cartan stochastic copying (or development) method. This method, which follows after Elworthy (1982), extends the one applied to classical trajectories (i.e., smooth curves) in previous articles (Rapoport, 1996a)[3] Namely, we copy on the configuration manifold, by "rolling without slipping" (i.e., keeping first-order contact) by parallel transport of a standard Wiener process (described by the Gaussian measure) on the homogeneous flat model space. $R^n$, which models the vacuum. In this way, we can generate the most general nonlinear diffusion on the configuration manifold starting from the simplest of all diffusion processes on the tangent space which is invariant by the orthogonal group. The whole

[2] Nonadherence to this principle leads to extremely cumbersome asymptotic solutions of diffusion processes with small "noise" terms (Gardiner, 1993).

[3] This method was developed to give a theory of a classical relativistic spinning test particle submitted to exterior Riemann–Cartan connections, which bypasses both Lagrangian and Hamiltonian structures (Rapoport and Sternberg, 1984a, b), and is founded on presymplectic geometry.

information for carrying this method rests on the RCW connection. In the case $M$ is a four-dimensional spacetime manifold (to make contact with general relativity), in the infinitesimal model $R^4$ ($R^{1,3}$) of the tangent space $T_xM$ at each point $x \in M$ seen as the quotient of the affine-orthogonal (Poincaré) group and the orthogonal (Lorentz) group, we have a standard Wiener diffusion whose probability measure is a Gaussian, and by rolling without slipping on spacetime by parallel transport by a RCW connection, we generate a generally non-Gaussian process on spacetime. We further remark that the prescription we shall follow for the diffusion process on $R^n$ is Stratonovich's midpoint prescription, which gives a stochastic calculus with the same transformation rules as the classical Cartan calculus on manifolds.

This article is organized as follows. In Section 1 we present the RCW geometries and the Fokker–Planck master operator from the point of view of giving a covariant representation of the master operator. In Section 2 we give a complete characterization of the Fokker–Planck operator and discuss the relation between the electromagnetic potentials appearing in the trace-torsion decomposition and the microscopic irreversibility of the quantum diffusions. In Section 3 we present the description of the quantum motions generated by the RCW Laplacian in terms of stochastic differential equations. In Section 4 we present the Cartan stochastic copying (or developing) method, as a final approach to the introduction of the RCW geometry. Finally, for the reader who is unacquainted with the theory of diffusions on manifolds and the stochastic calculi, we give an Appendix, in which we first give the basic notions, proceeding in the subsequent second section to the theory of stochastic differentials, and finally to the theory of diffusion processes on differentiable manifolds. This last section of the Appendix contains a theorem which is fundamental for the discussion of the Cartan stochastic development method.

## 1. THE INVARIANT FOKKER–PLANCK OPERATOR AND RIEMANN–CARTAN–WEYL GEOMETRIES

We shall always consider smooth $n$-dimensional, compact oriented manifolds $M$. We shall additionally assume given a second-order smooth differential operator $L$. On a local coordinate system $(x^\alpha)$, $\alpha = 1, \ldots, n$, we write $L$ as

$$L = \frac{1}{2} g^{\alpha\beta}(x)\partial_\alpha\partial_\beta + B^\alpha(x)\partial_\alpha \tag{1.1}$$

where $B^\alpha$, $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha$, $\beta = 1, \ldots, n$, are smooth functions on $M$.

From now on, we shall fix this coordinate system, and all local expressions shall be written in it. We shall be particularly interested in this section in the (backward) Fokker–Planck operator, given by the formal adjoint of $L$ with respect to the standard Hilbert space product of $L^2(R^n)$. We remark that in expression (1.1) we could have added a potential function. For probability-preserving semigroups, however, as we shall consider in the following, this function is automatically zero.

Although formally there is no restriction as to the nature of $M$, due to the group-theoretic foundations of Cartan's method we are working on a four-dimensional spacetime manifold (for formulating the relativistic theory), or a three-dimensional space manifold (for the nonrelativistic theory), and *not* in a phase-space manifold. Our presentation expands on Rapoport (1995c).

The *principal symbol* $\sigma$ of $L$ is the section of the bundle of real bilinear symmetric maps on $T^*M$, defined as follows: for $x \in M$, $p_i \in T_x^*M$, take $C^2$ functions, $f_i$: $M \to R$ with $f_i(x) = 0$ and $df_i(x) = p_i$, $i = 1, 2$; then,

$$\sigma(x)(p_1, p_2) = L(f_1 f_2)(x)$$

Note that $\sigma$ is well defined, i.e., it is independent of the choice of the functions $f_i$, $i = 1, 2$.

If $L$ is locally as in (1.1), then $\sigma$ is locally represented by the matrix $(g^{\alpha\beta})$. We can also view $\sigma$ as a section of the bundle of linear maps $L(T^*M, TM)$, or as a section of the bundle $TM \otimes TM$, or as a bundle morphism from $T^*M$ to $TM$. If $\sigma$ is a bundle isomorphism, it induces a metric $g$ on $M$, $g$: $M \to L(TM, TM)$:

$$g(x)(v_1, v_2) := \langle \sigma(x)^{-1} v_1, v_2 \rangle_x$$

for $x \in M$, $v_1$, $v_2 \in T_x^*M$. Here, $\langle .,. \rangle_x$ denotes the natural duality between $T_x^*M$ and $T_xM$. Locally, $g(x)$ is represented by the matrix $\frac{1}{2}(g^{\alpha\beta}(x))^{-1}$. Consider the quadratic forms over $M$ associated to $L$, defined as

$$Q_x(p_x) = \frac{1}{2} \langle p_x, \sigma_x(p_x) \rangle_x$$

for $x \in M$, $p_x \in T_x^*M$. Then, with the local representation (1.1) for $L$, $Q_x$ is represented as $\frac{1}{2}(g^{\alpha\beta}(x))$. Then, $L$ is an elliptic (semielliptic) operator whenever for all $x \in M$, $Q_x$ is positive definite (nonnegative definite). We shall assume in the following that $L$ is an elliptic operator. In this case, $\sigma$ is a bundle isomorphism and the metric $g$ is actually a Riemannian metric. Notice as well that $\sigma(df) = \text{grad} f$ for any $f$: $M \to R$ of class $C^2$, where grad denotes the Riemannian gradient.

In this section, we shall give an intrinsic description of $L$, i.e., a description independent of the local coordinate system. This is the essential prerequisite of covariance.

For this, we shall introduce an arbitrary connection on $M$, whose covariant derivative we shall denote as $\nabla$. We remark here that $\nabla$ need *not* be the

Levi-Civita connection associated to $g$; we shall make this precise below. Let $\sigma(\nabla)$ denote the second-order part of $L$, and let us denote by $X_0(\nabla)$ the vector field on $M$ given by the first-order part of $L$. Then, for $f: M \to R$ of class $C^2$, we have

$$\sigma(\nabla)(x) = \frac{1}{2}\,\mathrm{trace}(\nabla^2 f)(x) = \frac{1}{2}\,(\nabla df)(x) \tag{1.2}$$

where the trace is taken in terms of $g$, and $\nabla\, df$ is thought of as a section of $L(T^*M, T^*M)$. Also, $X_0(\nabla) = L - \sigma(\nabla)$. If $\Gamma^\alpha_{\beta\gamma}$ is the local representation for the Christoffel symbols of the connection, then the local representation of $\sigma(\nabla)$ is

$$\sigma(\nabla)(x) = \frac{1}{2}\,g^{\alpha\beta}(x)(\partial_\alpha\partial_\beta + \Gamma^\gamma_{\alpha\beta}(x)\partial_\gamma) \tag{1.3}$$

and

$$X_0(\nabla)(x) = B^\alpha(x)\partial_\alpha - \frac{1}{2}\,g^{\alpha\beta}(x)\Gamma^\gamma_{\alpha\beta}\partial_\gamma \tag{1.4}$$

If $\nabla$ is the Levi-Civita connection associated to $g$, which we shall denote as $\nabla^g$, then for any $f: M \to R$ of class $C^2$:

$$\sigma(\nabla^g)(df) = \tfrac{1}{2}\,\mathrm{tr}((\nabla^g)^2 f) = \tfrac{1}{2}\,\mathrm{tr}(\nabla^g df) = 1/2\ \mathrm{div}_g\,\mathrm{grad}\,f = 1/2\Delta_g f \tag{1.5}$$

Here, $\nabla_g$ is the Levi-Civita Laplacian operator on functions; locally, it is written as

$$\Delta_g = g^{-1/2}\partial_\alpha((g^{1/2}g^{\alpha\beta}\partial_\beta); \qquad g = \det(g_{\alpha\beta}) \tag{1.6}$$

and $\mathrm{div}_g$ is the Riemannian divergence operator on vector fields $X = X^\alpha(x)\partial_\alpha$:

$$\mathrm{div}_g(X) = g^{-1/2}\partial_\alpha(g^{1/2}X^\alpha) \tag{1.7}$$

We now take $\nabla$ to be a Riemann–Cartan connection with torsion, which we additionally assume to be compatible with $g$, i.e., $\nabla g = 0$. Then $\sigma(\nabla) = \frac{1}{2}\mathrm{tr}(\nabla^2)$. Let us compute this. Denote the Christoffel coefficients of $\nabla$ as $\Gamma^\alpha_{\beta\gamma}$; then,

$$\Gamma^\alpha_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + 1/2K^\alpha_{\beta\gamma} \tag{1.8}$$

where the first term in (1.8) stands for the Christoffel Levi-Civita coefficients of the metric $g$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta} \tag{1.9}$$

is the cotorsion tensor, with $S^{\alpha}_{\beta\gamma} = g^{\alpha\nu}g_{\beta\kappa}T^{\kappa}_{\nu\gamma}$, and $T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$ the skew-symmetric torsion tensor.

Let us consider (one-half) the Laplacian operator associated to this Cartan connection, defined—extending the usual definition—by

$$H(\nabla) = 1/2 g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta} \tag{1.10}$$

Then, $\sigma(\nabla) = H(\Gamma)$. A straightforward computation shows that that $H(\nabla)$ only depends on the trace of the torsion tensor and $g$:

$$H(\nabla) = 1/2\Delta_g + g^{\alpha\beta}Q_{\beta}\partial_{\alpha} \tag{1.11}$$

with

$$Q = T^{\nu}_{\nu\beta}\, dx^{\beta}$$

which is the trace-torsion one-form.

Therefore, for the Riemann–Cartan connection $\nabla$ defined in (1.8), we have that

$$\sigma(\nabla) = \frac{1}{2}\operatorname{tr}(\nabla^2) = \frac{1}{2}\Delta_g + \hat{Q} \tag{1.10$'$}$$

with $\hat{Q}$ the vector field conjugate to the 1-form $Q$: $\hat{Q}(f) = \langle Q, df\rangle$, $f\colon M \to R$. In local coordinates,

$$\hat{Q}^{\alpha} = g^{\alpha\beta}Q_{\beta}$$

We further have

$$X_0(\nabla) = B - \frac{1}{2}g^{\alpha\beta}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}\partial_{\gamma} - \hat{Q} \tag{1.12}$$

Therefore, the invariant decomposition of $L$ is

$$\frac{1}{2}\operatorname{tr}(\nabla^2) + X_0(\nabla) = \frac{1}{2}\Delta_g + b \tag{1.13}$$

with

$$b = B - \frac{1}{2}g^{\alpha\beta}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}\partial_{\gamma} \tag{1.14}$$

Notice that (1.12) can be thought of as arising from the 'gauge' transformation $\tilde{b} \to \tilde{b} - Q$, with $\tilde{b}$ the 1-form conjugate to $b$.

If we take for a start $\nabla$ with Christoffel symbols of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix} + \frac{2}{n-1}\{\delta^{\alpha}_{\beta}Q_{\gamma} - g_{\beta\gamma}Q^{\alpha}\} \tag{1.15}$$

with

$$Q = \tilde{b}, \qquad \text{i.e.,} \qquad \hat{Q} = b \tag{1.16}$$

we have

$$X_0(\nabla) = 0 \tag{1.17}$$

and

$$\sigma(\nabla) = \frac{1}{2}\,\mathrm{tr}(\nabla^2) = H(\nabla) = \frac{1}{2}\,\mathrm{tr}((\nabla^g)^2) + \hat{Q} = \frac{1}{2}\,\Delta_g + b \tag{1.18}$$

Therefore,

$$L = \sigma(\nabla) = H(\nabla) = \frac{1}{2}\,\mathrm{tr}((\nabla^g)^2) + \hat{Q} \tag{1.19}$$

The choice of the RCW connection $\nabla$ is now clear: its Laplacian contains both the second-order and first-order terms in a single covariant operator. Furthermore, if we rescale $\nabla$ and consider instead $\hbar\nabla$, then we observe that both terms in (1.19) are quadratic in $\hbar$.

The restriction we have placed on $\nabla$ to be as in (1.15), i.e., only the trace component of the irreducible decomposition of the torsion tensor is taken, is due to the fact that all other components of this tensor do not appear at all in the Laplacian of (the otherwise too general) $\nabla$. In the particular case of dimension 2, this is automatically satisfied. In the case we actually have assumed, that $g$ is Riemannian, the expression (1.19) is the most general invariant Laplacian acting on functions defined on a smooth manifold. This restriction will allow us to establish a one-to-one correspondence between Riemann–Cartan connections of the form (1.15) and Markovian diffusion processes. We called these metric-compatible connections RCW (Riemann–Cartan–Weyl) geometries since the trace-torsion is a Weyl 1-form (Rapoport, 1991, 1996a). Thus, these geometries do not have the historicity problem which led to Einstein's rejection of the first gauge theory, that proposed by Weyl.

## 2. THE WEYL TRACE-TORSION ONE-FORM

### 2.1. The De Rham–Kodaira–Hodge Decomposition of the Weyl Form and the Maxwell–de Rham Equations of Electromagnetism

To obtain the most general form of the RCW Laplacian, we need to determine the most general form of a 1-form on a manifold provided with a metric, to further apply it to the trace-torsion 1-form. The answer to this problem, for an oriented compact manifold is given by the well-known de Rham–Kodaira–Hodge theorem (de Rham, 1984), which we recall now.

We consider the Hilbert space of square-summable ω of smooth differential forms of degree $q$ on $M$ with respect to $\mathrm{vol}_g$. We shall denote this space as $L^{2,q}$, or as $L^2\Omega^q(M, \mathrm{vol}_g)$. The inner product is

$$\langle \omega, \phi \rangle = \int_M \langle \omega(x), \phi(x) \rangle \, \mathrm{vol}_g \tag{2.1}$$

where the integrand is given by the natural pairing between the components of ω and the conjugate tensor: $g^{\alpha_1\beta_1} \cdots g^{\alpha_q\beta_q}\phi_{\beta_i \cdots \beta_q}$; alternatively, we can write in a coordinate-independent way: $\langle \omega(x), \phi(x) \rangle \, \mathrm{vol}_g = \omega(x) \wedge * \, \phi(x)$, with $*$ the Hodge star operator, for any $\omega, \phi \in L^{2,q}$.

The de Rham–Kodaira operator on $L^{2,q}$ is defined as

$$\Delta = (d - \delta)^2 = -(d\delta + \delta d) \tag{2.2}$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator $d$ defined on $L^{2,q}$:

$$\langle \delta\phi, \omega \rangle = \langle \phi, d\omega \rangle$$

for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$. Then,

$$\delta^2 = 0$$

In the case of $q = 0$, the de Rham–Kodaira operator coincides with the Laplace–Beltrami operator on functions encountered before; in the general case we have in addition to $\mathrm{tr}(\nabla^g)^2$ the contribution of the Weitzenbock curvature term, which we shall describe below.

The de Rham–Kodaira–Hodge theorem states that $L^{2,1}$ admits the following invariant decomposition. Let $\omega \in L^{2,1}$; then,

$$\omega = df + A_{\mathrm{cocl}} + A_{\mathrm{harm}} \tag{2.3}$$

where $f: M \to R$ is a smooth function on $M$, $A_{\mathrm{cocl}}$ is a coclosed smooth 1-form,

$$\delta A_{\mathrm{cocl}} = -\mathrm{div}_g \hat{A}_{\mathrm{cocl}} = 0$$

and $A_{\mathrm{harm}}$ is a coclosed and closed smooth 1-form:

$$\delta A_{\mathrm{harm}} = 0 \qquad dA_{\mathrm{harm}} = 0 \tag{2.4}$$

Otherwise stated, $A_{\mathrm{harm}}$ is a harmonic one-form, i.e.,

$$\Delta A_{\mathrm{harm}} = 0 \tag{2.5}$$

Furthermore, this decomposition is orthogonal in $L^{2,1}$, i.e.,

$$\langle df, A_{\mathrm{cocl}} \rangle = \langle df, A_{\mathrm{harm}} \rangle = \langle A_{\mathrm{cocl}}, A_{\mathrm{harm}} \rangle = 0 \tag{2.6}$$

Now we return to the problem of characterizing the most general form of the trace torsion one-form $Q$. By the de Rham–Kodaira–Hodge theorem, there exists a smooth function $f$ defined on $M$, and one-forms $A_{\mathrm{cocl}}$, $A_{\mathrm{harm}}$ on $M$ which are coclosed and harmonic, respectively, and such that $df$, $A_{\mathrm{cocl}}$, and $A_{\mathrm{harm}}$ are orthogonal in $L^{2,1}$, then the trace-torsion of the connection $\nabla$ is

$$Q = df + A_{\mathrm{cocl}} + A_{\mathrm{harm}} \tag{2.7}$$

We are interested in Markovian semigroups[4] $\{P_\tau, \tau \geq 0\}$ with infinitesimal generator given by $H(\nabla)$:

$$H(\nabla)h = \text{str}\lim_{\tau\to 0} \frac{P_\tau h - h}{\tau}$$

for $h$ in the domain of H($\nabla$); here, the limit is taken in the strong (operator) sense. Note that trivially $\{P_\tau, \tau \geq 0\}$ preserves probability, i.e., $P_\tau(1) = 1$ for any $\tau \geq 0$, i.e., the zeroth order ("potential") term which could have been included in (1.1) is automatically zero. Thus, we shall be interested in a probability-preserving semigroup whose covariant infinitesimal generator is $H(\nabla)$ described in (1.19), (2.7), i.e., one-half the RCW Laplacian operator.

We must remark that $\tau$ is not to be confused with the relativistic time coordinate of $M$; it is to be thought of as an internal time evolution parameter of the diffusion, which we shall describe below, consistently with our introduction of the master equation; this distinction between $\tau$ and the relativistic time coordinate was originally conceived in quantum gravity by B. de Witt. This time parameter is Liouville's time in Prigogine's theory of nonequilibrium statistical mechanics (Prigogine, 1962, 1995).

Therefore, the most general invariant Laplacian $H(\nabla)$ on a compact smooth manifold depends on $g$, $f$, $A_{\mathrm{cocl}}$, and $A_{\mathrm{harm}}$; thus we write

$$H(\nabla) = H(g, f, A_{\mathrm{cocl}}, A_{\mathrm{harm}})$$

Then acting on functions $\phi$ defined on $M$, we have

$$H(g, f, A_{\mathrm{cocl}}, A_{\mathrm{harm}})\phi = 1/2\Delta_g\phi + g(df, d\phi) + g(A_{\mathrm{cocl}}, d\phi) + g(A_{\mathrm{harm}}, d\phi) \tag{2.8}$$

or in local coordinates,

$$H(g, f, A_{\mathrm{cocl}}, A_{\mathrm{harm}})\phi = 1/2g^{\alpha\beta}(\partial_\beta f + A_\beta)\partial_\alpha\phi$$

[4] See Gardiner (1993) and Yosida (1980). A Markovian semigroup in a Hilbert space $H$ is a family of bounded, positive, linear operators $\{P_\tau, \tau \geq 0\}$ with dense domain contained in $H$, such that $P_0 = Id$ satisfying the properties (i) (semigroup property) $P_\tau \circ P_{\tau'} = P_{\tau+\tau'}$, $\tau, \tau' \geq 0$, (ii) (contraction property) $\|P_\tau\| \leq 1$, $\tau \geq 0$, and (iii) $\tau \to P_\tau$, is strongly continuous.

with

$$A = (A_{\text{cocl}} + A_{\text{harm}})$$

This is the (forward) Fokker–Planck operator of our theory.

The role of the trace-torsion vector field

$$b = \hat{Q} = \operatorname{grad} f + \hat{A}_{\text{cocl}}\, \hat{A}_{\text{harm}} \tag{2.9}$$

which is the vector field conjugate to the trace-torsion 1-form, is that of the drift (average velocity) of the continuous sample curves of the diffusion processes associated with $H(g, f, A_{\text{cocl}}, A_{\text{harm}})$. Thus, the introduction of the torsion is an essential feature of the diffusion processes associated with $\{P_\tau, \tau \geq 0\}$, since Brownian processes have continuous nondifferentiable sample paths (actually, they are fractals).

We are interested now in the transpose of $H(g, f, A_{\text{cocl}}, A_{\text{harm}})$ in $L^{2,0}$, i.e., the operator acting on smooth functions $\phi$ on $M$:

$$H(g, f, A_{\text{cocl}}, A_{\text{harm}})^{\dagger}\phi = \frac{1}{2}\,\Delta_g\phi - \operatorname{div}_g(\phi \cdot \operatorname{grad} f) - \operatorname{div}_g(\phi \cdot \hat{A}) \tag{2.10}$$

where we have kept the above notation, so that $\hat{A}$ is the vector field conjugate to $A$. The operator described by (2.10) is the (backward) Fokker–Planck operator (Gardiner, 1993).

The transition density $p_\tau^\nabla(x, y)$ is determined as the fundamental solution of the "heat" (here, for well-posedness, $g$ cannot be Lorentzian) equation on the first variable $x$:

$$\frac{\partial p_\tau^\nabla(x, y)}{\partial \tau} = H(g, f, A_{\text{cocl}}, A_{\text{harm}})(x)p_\tau^\nabla(x, y) \tag{2.11}$$

It will be very important for the following to note that the semigroup $\{P_\tau: \tau \geq 0\}$ has a unique $\tau$-independent *invariant* probability density $\rho > 0$ determined as the weak fundamental solution (in the sense of the theory of generalized functions) of the stationary ($\tau$-independent) Fokker–Planck equation:

$$H(g, f, A_{\text{cocl}}, A_{\text{harm}})^{\dagger}(\rho) = 0 \tag{2.12}$$

Let us determine $Q$ from (2.12). We choose a smooth function $U$ defined on $M$ such that

$$H(g, f, A_{\text{cocl}}, A_{\text{harm}})^{\dagger}(e^{-U}) = 0$$

i.e., $\rho = e^{-U} \operatorname{vol}_g$ is an invariant measure. Since

$$H(g, f, A_{\text{cocl}}, A_{\text{harm}})^\dagger(e^{-U}) = -1/2\delta d(e^{-U}) + \delta(e^{-U}Q)$$

$$= \frac{-1}{2}\delta d(e^{-U}) + \delta(e^{-U}(df + A_{\text{cocl}} + A_{\text{harm}})) = 0$$

i.e.,

$$-1/2\delta(d(e^{-U}) + e^{-U}[df + A_{\text{cocl}} + A_{\text{harm}}]) = 0$$

we finally have

$$e^{-U}(df + A_{\text{cocl}} + A_{\text{harm}}) - \frac{1}{2} d(e^{-U}) = \delta\beta_2 + \omega_1 \tag{2.13}$$

for some two-form $\beta_2$, and $\omega_1$ a harmonic one-form, i.e.,

$$e^{-U}(df + A_{\text{cocl}} + A_{\text{harm}}) = \frac{1}{2} d(e^{-U}) + \delta\beta_2 + \omega_1 \tag{2.14}$$

Therefore, if we write

$$\rho = \psi^2 \operatorname{vol}_g \qquad \text{i.e.,} \qquad U = -\ln \psi^2 \tag{2.15}$$

then the exact term of $Q$ is

$$df = d \ln \psi \tag{2.15}$$

and also we have the coclosed 1-form

$$A_{\text{cocl}} = \frac{\delta\beta_2}{\psi^2} \tag{2.16}$$

where $\beta_2$ is a smooth two-form on $M$, and

$$A_{\text{harm}} = \frac{\omega_{\text{harm}}}{\psi^2} \tag{2.17}$$

is a harmonic one-form on $M$ such that $\omega_{\text{harm}}$ is also a harmonic one-form on $M$; therefore $A_{\text{harm}}$ and $\omega_{\text{harm}}$ satisfy (2.4), or what is the same, satisfy (2.5). We remark that $d \ln \psi$, $A_{\text{harm}}$, and $A_{\text{cocl}}$ are chosen orthogonal in $L^{2,1}$. Therefore we shall write the forward Fokker–Planck operator as $H(g, \psi, A_{\text{cocl}}, A_{\text{harm}})$

Conversely, if $Q$ is described by the sum of (2.15)–(2.17), then

$$H(g, \psi, A_{\text{cocl}}, A_{\text{harm}})^\dagger(e^{-U}) = \delta(e^{-U}(d \ln \psi + A_{\text{cocl}} + A_{\text{harm}}) - \frac{1}{2} d(e^{-U}))$$

$$= \delta(\delta\beta_2 + \omega_{\text{harm}}) = 0$$

and hence $\psi^2 \mathrm{vol}_g$ is an invariant measure; here we have used the fact that $\delta^2 = 0$ and $\omega_{\mathrm{harm}}$ is coclosed.

Therefore, from the stationary Fokker–Planck equation we have determined the de Rham–Kodaira–Hodge decomposition of the trace torsion one-form $Q$; it is given by the sum of exact, coclosed, and harmonic 1-forms:

$$Q = d \ln \psi + 1/\psi^2(\delta\beta_2 + \omega_{\mathrm{harm}}) \tag{2.18}$$

with $\beta_2$ a smooth two-form, and $\omega_{\mathrm{harm}}$ a harmonic one-form, both defined on $M$. This is the *orthogonal* decomposition of the trace-torsion one-form $Q$.

The exact term of $Q$ is the one that appears upon introducing the Einstein $\Lambda$ transformations produced by the $\psi$-field (Rapoport, 1991, 1996a), which we proved (Rapoport *et al.*, 1994) coincides to the amplitude of a Dirac–Hestenes spinor operator field. Therefore, we conclude that associated to the $\Lambda$ transformations by $\psi$, there exists an invariant stationary probability density $\psi^2\mathrm{vol}_g$. From the point of view of statistical mechanics this density plays the role of a Gibbs measure (Graham and Haken, 1971; Rapoport, 1996c, d; Nagasawa, 1993), while in (nonrelativistic) quantum mechanics it is the Born density. Furthermore, through the determination of the form of the electromagnetic potentials appearing in the decomposition of the trace-torsion one-form, we have found that they appear normalized by $\psi^2$; similarly to the modification of the electromagnetic potential found by Hojman *et al.* (1979) in studying the problem of coupling of electromagnetism to the exact component of $Q$.

It is important to remark that $\rho$ describes the *final* state of the system. Indeed, one can prove that the transition density $p_\tau^\nabla(x, y)$ tends to $\rho(y)$ for $\tau \to \infty$, $y \in M$, and $x$ in a compact set.

It is quite remarkable that the condition of existence of a stationary solution of the Fokker–Planck equation (2.9) leads to a decomposition of the trace-torsion in which there appear two potentials, one of which is harmonic, which are further normalized by $1/\psi^2$. We further note that if we take $\psi$ such that it has a nonnull node set $N(\psi) = \{x \in M: \psi(x) = 0\}$, then these two potentials, $\delta\beta_2/\psi^2$ and $\omega_{\mathrm{harm}}/\psi^2$, become singular at $N(\psi)$.

The electromagnetic field of the coclosed potential is

$$F = d\left(\frac{\delta\beta_2}{\psi^2}\right) = \frac{1}{\psi^2}\, d\delta\beta_2 + d\left(\frac{1}{\psi^2}\right) \wedge \delta\beta_2$$

$$= 1/\psi^2(d\delta\beta_2 - 2d \ln \psi \wedge \delta\beta_2) \tag{2.19}$$

Then, $F$ satisfies the Maxwell equations

$$dF = 0 \tag{2.20}$$

and

$$\delta F = j \tag{2.21}$$

where $j$ is the electric current 1-form. In taking into account that $\delta A_{\text{cocl}} = 0$, we can rewrite (2.21) as

$$\delta F = \delta d A_{\text{cocl}} = -\Delta A_{\text{cocl}} = j \tag{2.22}$$

or as

$$\Delta A_{\text{cocl}} = \text{trace}((\nabla^g)^2) A_{\text{cocl}} - R^\beta_\alpha(g) A_{\text{cocl}\beta} dx^\alpha = -j \tag{2.23}$$

with

$$R^\beta_\alpha(g) = R_{\mu\alpha}{}^{\mu\beta}(g)$$

which is the Ricci curvature tensor associated to $g$. This is the source-full Maxwell–de Rham equation. As observed by Misner *et al.* (1973, p. 569), these equations are more general (and also intrinsic) than the ones obtained from the application of the principle of equivalence, by replacing usual derivatives by Levi-Civita connection derivatives, and reduces to them when spacetime is flat. Indeed, in (2.23) there is a coupling to curvature, which is called the Weitzenbock term.

Let us examine now the harmonic term $A_{\text{harm}} = \omega_{\text{harm}}/\psi^2$ of $Q$. Trivially, it generates a trivial electromagnetic field, so that we shall call it the Aharonov–Bohm potential term of $Q$. The fact that it is coclosed can be expressed in the form of the conservation equation (Lorentz gauge condition)

$$\text{div}_g \hat{A}_{\text{harm}} = -\delta A_{\text{harm}} = 0 \tag{2.24}$$

Yet the fact that $A_{\text{harm}}$ is harmonic can be written in the form of the sourceless Maxwell–de Rham equation,

$$\Delta A_{\text{harm}} = 0 \tag{2.25}$$

Furthermore, if we let $\psi$ have a nonnull node set on $M$, on this set $A_{\text{harm}}$ becomes singular, and then it is natural to generalize (2.25) to an inhomogeneous equation of the form

$$\Delta A_{\text{harm}} = j_{\text{mag}} \tag{2.25'}$$

where $j_{\text{mag}}$ denotes a magnetic monopole current one-form.

## 2.2. Irreversibility and the Nonexact Weyl One-Form

Finally, we shall introduce the probability vector associated to the RCW diffusion. Consider the vector field

$$J_\tau := p^\nabla_\tau b - \frac{1}{2} \text{grad}\, p^\nabla_\tau \tag{2.26}$$

Then the Fokker–Planck equation can be rewritten as

$$\frac{\partial p_\tau^\nabla}{\partial \tau} + \mathrm{div}_g J_\tau = 0 \tag{2.27}$$

which in the stationary state yields the probability vector field

$$J_{\mathrm{stat}} = \psi^2 b - \frac{1}{2}\,\mathrm{grad}\,\psi^2 \tag{2.28}$$

where, as above, $b$ is the conjugate vector field to $Q$ given by (2.18), i.e.,

$$b = \mathrm{grad}\ln\psi + \hat{A}/\psi^2 \tag{2.29}$$

or, in local coordinates,

$$b^\alpha = g^{\alpha\beta}(\partial_\alpha \ln\psi + A_\beta/\psi^2) \tag{2.29'}$$

with $A$ given by the coclosed 1-form

$$A = \delta\beta_2 + \omega_{\mathrm{harm}}$$

Therefore, $J_{\mathrm{stat}}$ reduces to

$$J_{\mathrm{stat}} = \hat{A} \tag{2.30}$$

Then, $J_{\mathrm{stat}}$ is a conserved probability vector field, since

$$\mathrm{div}_g J_{\mathrm{stat}} = -\delta J_{\mathrm{stat}} = -\delta\hat{A} = 0 \tag{2.31}$$

Note that only in the case in which we set $A \equiv 0$, or equivalently, by orthogonality, that $\delta\beta_2$ and $\omega_{\mathrm{harm}}$ both vanish, do we have a null probability vector. This will be of importance in relation to the microscopic reversibility of the quantum fluctuations, which we shall introduce in terms of detailed balance. A nonnull probability vector field turns out to be equivalent to irreversibility of the quantum fluctuations.

Indeed, the irreversibility of diffusions [i.e., that the integral quantum flow $\{X_\tau: \tau \geq 0\}$ generated by $H(\nabla)$ is probabilistically distinguishable from the time-reversed process $\{X_{-\tau}: \tau \geq 0\}$] is usually presented in terms of the notion of detailed balance in nonequilibrium thermodynamics (Graham and Haken, 1971; Van Kampen, 1957; Gardiner, 1993; Rapoport, 1996c). There is a simple characterization of irreversibility in terms of the infinitesimal generator of a diffusion process and its stationary density: The diffusion process is reversible iff the drift vector field reduces to the exact component; the noncovariant formulation of this is due to Graham and Haken (1971). This characterization of the irreversibility of a diffusion process can be given covariantly in terms of the lack of symmetricity of its infinitesimal generator. A diffusion process with infinitesimal generator $L$ and stationary density $\mu$

is symmetric (i.e., reversible) iff the operator $L$ is symmetric on the Hilbert space $L^2(\mu)$ (Kolmogorov, 1937; Nagasawa, 1993). Thus, reversibility is satisfied whenever, for any two smooth functions $f$ and $g$ defined on $M$, we have that

$$\int (Lf)(x)g(x)\ \mu(dx) = \int f(x)(Lg)(x)\ \mu(dx)$$

It is straightforward to check that for RCW diffusion with generator $H(\nabla) = H(g, \psi, A_{\text{cocl}}, A_{\text{harm}})$ and stationary measure $\psi^2\text{vol}_g$, the process is reversible if and only if $\delta\beta_2$ and $\omega_{\text{harm}}$ vanish completely, i.e., the probability vector field in the stationary state $J_{\text{stat}}$ vanishes identically. Therefore, the nonexact terms of $Q$, which thus cannot be gauged away, appear associated to the breaking of irreversibility of the spin-0 diffusions generated by the RCW connection.

## 3. RIEMANN–CARTAN–WEYL GEOMETRIES AND STOCHASTIC MOTIONS

We have already described the geometrical structures which lead to the Fokker–Planck irreversible equations of diffusion determined by these geometries. Now, as is well known in a noncovariant setting, the Fokker–Planck equations are equivalent to stochastic differential equations (Gardiner, 1993), whose covariant description we give now. The description of the geometrically determined fluctuations of spacetime described by these stochastic equations can be given in essentially two ways; the first, which we describe below, demands the introduction of an arbitrary "square root" of the metric. This construction is nonunique, geometrically speaking, yet probabilistically it is essentially unique (see Appendix). In Section 4 we shall give a construction which retains essentially the meaning of torsion, as described by Cartan's development method, and does not demand the "square root" of the metric.

By embedding $M$ on $R^d$, with $d \leq 2n + 1$, we can obtain a section $Y$ (at least locally Lipschitz, or satisfying the Sobolev regularity conditions) of $L(R^d, TM)$, so that if $Y^*$ denotes the dual section of $L(TM, R^d)$, then, for all $x \in M$,

$$g(x) = Y(x)Y^*(x) \tag{3.1}$$

Given an orthonormal basis $\{e_i, i = 1, \ldots, n\}$ of $R^d$, we may define vector fields

$$Y_i(x) = Y(x)(e_i) \tag{3.2}$$

Taking $Y$ to be smooth, we can define the second-order differential operator

$$L_Y^2 = (Y_i)^2 \tag{3.3}$$

For $L$ as in (1.1) and $Y$ locally of the form

$$Y_i(x) = Y_\beta^\alpha(x)\partial_\alpha \tag{3.4}$$

we have, locally,

$$L_Y^2(x) = g^{\alpha\beta}\partial_\alpha\partial_\beta + Y_\alpha^\beta(x)\partial_\beta Y_\alpha^\gamma(x)\partial_\gamma \tag{3.5}$$

If we take the vector field on $M$ given by

$$X_0^Y(x) = B(x) - \frac{1}{2} Y_\alpha^\beta\partial_\beta Y_\alpha^\gamma(x)\partial_\gamma \tag{3.6}$$

then

$$L = \frac{1}{2} L_Y^2 + X_0^Y \tag{3.7}$$

This decomposition depends essentially on the choice of the "square root" $Y$ of $\sigma$.

For an arbitrary Riemann–Cartan connection $\nabla$ as in (1.15), we consider its term given by the Levi-Civita connection $\nabla^g$; with the choice of $Y$ sufficiently regular, we have the following decomposition of $L = H(\nabla)$ as in (3.7):

$$X_0^Y = X_0(\nabla) - S(\nabla^g, Y) \tag{3.8}$$

and

$$\frac{1}{2} L_Y^2 = \sigma(\nabla) + S(\nabla^g, Y) \tag{3.9}$$

with

$$S(\nabla^g, Y) = \frac{1}{2} \operatorname{tr}(\nabla^g Y(Y(\cdot)(\cdot)) \tag{3.10}$$

where $\nabla^g Y$ is the covariant derivative of $Y$ viewed as a section of $L(TM, L(R^d, TM))$. The vector field $S(\nabla^g, Y)$ is the Stratonovich correction term. It is essential for the transformation of the representation of the process according to the Stratonovich midpoint rule, which gives the Stratonovich stochastic differential calculus with transformation rules similar to the ordinary differential calculus, to the Itô representation (prepoint rule).

If we take for $\nabla$ the RCW connection (1.15), since $\sigma(\nabla) = 1/2\ \operatorname{tr}(\nabla^2) = H(g, \psi, A_{\text{cocl}}, A_{\text{harm}})$, then we obtain from (1.17) and (3.8)

$$X_0^Y = -S(\nabla^g, Y) \tag{3.11}$$

and

$$\frac{1}{2} L_Y^2 = \frac{1}{2} \Delta_g + b + S(\nabla^g, Y), \qquad \text{with } b = \hat{Q} \tag{3.12}$$

Given a Fokker–Planck operator, one can characterize the associated Markovian semigroup through the heat kernel or through a stochastic differential equation; for the noncovariant formulation see Gardiner, (1993). In particular, for the Markovian semigroups determined by $H(g, \psi, A_1, A_2)$, the associated stochastic differential equation for the random continuous curves $\{X_\tau: \tau \geq 0\}$ is

$$dX_\tau = Y(X_\tau) \circ dW_\tau + [b(X_\tau) + S(\nabla^g, Y)](X_\tau)\, d\tau \tag{3.13}$$

in the Stratonovich representation, or equivalently, in the form of the Itô representation,

$$dX_\tau = Y(X_\tau)\, dW_\tau + b(X_\tau)\, d\tau \tag{3.14}$$

with $b$ given by (2.29) and $\{W_\tau, \tau \geq 0\}$ is a mean-0 Brownian process on $R^d$: $E(W_\tau) = 0$ and $E(W_\tau^r W_\tau^s) = \delta^{rs}\tau$, the Krönecker tensor.

## 4. THE CARTAN STOCHASTIC COPYING METHOD

We shall take $M$ to be a smooth $n$-dimensional manifold (which can be in particular a four-dimensional spacetime) provided with a Riemann–Cartan structure $\nabla$ which is compatible with the Riemannian metric $g$. On $M$ we construct the bundle $\pi: P_H \to M$ of orthogonal frames. Thus, consistent with Section 1 of Rapoport (1996a) (which we shall denote from now on as I), $g$ is a Riemannian metric and $H$ is the orthogonal group $0(n)$ as isometry group of the tangent space at each point of $M$, $R^n$ provided with the flat metric diag$(1, \ldots, 1)$. It is important to remark that $g$ can be in fact a Lorentzian metric on $M$, so that in this case $H = SO_+(1, n - 1)$ and instead of $R^n$ we take $R^{1,n-1}$, the Minkowski space provided with the metric diag$(+1, -1, \ldots, -1)$. We shall take the canonical realization of the standard Wiener process $W(\tau, \omega) = \omega(\tau)$ on $R^n(R^{1,n-1})$; we have $E(W^a) = 0$ and $E(W_\tau^a W_\tau^b) = \delta_b^a \tau$ for all $a, b \in \{1, \ldots, n\}$.

The Cartan stochastic copying method is embodied in the following stochastic differential equation, which replaces equations (7.5), (7.6) of I.

We look for the solution $r(\tau) = (r(\tau, r, w))$, where $(r(\tau)) \in P_H$, of the Stratonovich stochastic differential equation

$$dr(\tau) = \tilde{L}_a(r(\tau)) \circ dw^a(\tau) \tag{4.1}$$

with initial condition

$$r(0) = r \tag{4.2}$$

Here, $\tilde{L}_a$, $a = 1, \ldots, n$, are the canonical horizontal vector fields associated with the metric compatible connection $\nabla$, given by

$$\tilde{L}_a = e_a^\alpha \partial_\alpha - \Gamma^\delta_{\beta\gamma} e_a^\beta e_b^\gamma \frac{\partial}{\partial e_b^\delta} \tag{4.3}$$

In local coordinates of $P_H$, equations (4.1), (4.2) read

$$dX^\alpha(\tau) = e_a^\alpha(\tau) \circ dw^a(\tau) \tag{4.4}$$

$$de_a^\alpha(\tau) = -\Gamma^\alpha_{\beta\gamma}(X(\tau)) e_a^\beta(t) \circ dX^\alpha(\tau) \tag{4.5}$$

where $r(\tau) = (X^\alpha(\tau), e_a^\alpha(\tau))$. Note that if $r(0) \in P_H$, then the whole solution $r(\tau)$ remains in $P_H$, since, as can be verified straightforwardly, it is valid that

$$d(g_{\alpha\beta}(X(\tau) e_a^\alpha(\tau) e_b^\beta(\tau)) = 0 \tag{4.6}$$

so that $\nabla$ is compatible with $g$ along $X(\tau)$. (This is quite remarkable, since $X_\tau$ is only continuous.)

Now a stochastic curve on $M$ is defined just as in the classical case by projection on $M$: $X(\tau) = \pi(r(\tau))$. By writing $X(\tau) = (X(\tau, r, w))$, we have, just as in the classical case presented in I, that

$$X(\tau, Ar, w) = X(\tau, r, Aw), \qquad \forall \tau \geq 0, \qquad A \in H, w \tag{4.7}$$

But $Aw = (Aw(\tau))$ is another Wiener process; this is an essential property of the Wiener process, its invariance by the (pseudo) orthogonal group.[5] Hence, the probability law of $X(., Ar, w)$ is independent of $A \in H$. Then the probability law of $X(., r, w)$ depends only on $x = \pi(r)$; we denote it by $P_x$. Thus we have a diffusion process on $M$; in fact it can be proved that it is strong Markov diffusion (for these notions, see Appendix). Now, let us compute the infinitesimal generator $A$ of this Brownian process. For any function $h$ on $P_H$, $h(r) \equiv h(x)$, for $r = (x, e)$ of class $C^3$ (so that $h$ is a "basic" function), we have by integrating (4.1) with (4.2)

$$h(X_\tau) - h(x) = \int_0^\tau \tilde{L}_a(r(\sigma)) \circ dw^a(\sigma) \tag{4.8}$$

Now we apply Theorem A.1 (see Appendix). In the present case $A_0 \equiv 0$ and the $A_\alpha$ are given by (4.3). Then, from Theorem A.1 we conclude that the infinitesimal generator $A$ of the diffusion $X(\tau) = \pi(r(\tau))$ on $M$ is

$$Af = 1/2 \tilde{L}_a \tilde{L}_a f \tag{4.9}$$

[5] Remarkably, the Wiener process is invariant by the action of the whole 15-dimensional conformal group, yet, under the action of the inversion transformations $J(x) = x/|x|^2$, it is actually an $h$-transform of the Wiener process, where $h = |x|^{2-n}$, as a function on $R^n$.

for any smooth function $f$ on $M$. Now, a straightforward computation yields

$$Af = 1/2 g^{\alpha\beta}\nabla_\alpha \nabla_\beta f = H(\nabla) \tag{4.10}$$

It is essential to remark that this operator is one-half the Laplacian operator (1.10) associated with the metric-compatible connection $\nabla$.

Thus, we have completed the presentation of the extension of Cartan's method to Wiener processes. It is remarkable here, in contrast with the presentation of Section 3, that there is no need of a square root of the metric. The metric compability appears as the condition of classical reduction (Rapoport and Sternberg, 1984a, b; Rapoport 1996a) of the linear bundle of frames to the orthogonal (or Lorentz) bundle of frames $P_H$, which itself allows the definition of the stochastic process on $M$ as the projection of the stochastic process on $P_H$. We remark that to ensure a one-to-one correspondence between Riemann–Cartan connections and stochastic processes with infinitesimal generator given by $H(\nabla)$, we need to restrict $\nabla$ to be a Riemann–Cartan–Weyl connection, which by the same arguments as presented in Section 1, takes the form given in (1.15), with trace-torsion Weyl one-form given in (2.18).

## 5. CONCLUSIONS

We have introduced the RCW geometries and their Laplacian operator on functions from two different points of view. For the first one, starting from a noncovariant Fokker–Planck operator, we have seen that the covariant decomposition of this operator leads naturally to the RCW geometries.

Second, starting from the simplest of all diffusion processes, the canonical Wiener process with null drift, we have obtained by Cartan's method the most general probability-conserving Markovian semigroup in spacetime. All of the information for carrying out this *stochastic Cartan method* is incorporated in the Riemann–Cartan–Weyl connection. While infinitesimally, on the tangent space we have the trivial Wiener measure corresponding to a reversible process, on spacetime we have a generally irreversible process which is no longer described by a Gaussian measure. The processes constructed by the Cartan stochastic copying method are indistinguishable from those constructed in Section 3 by taking a square root of the metric to describe the fluctuations. Remarkably, these constructions can be extended to infinite variables, in considering Gibbsian measures on lattices given by infinite denumerable products of $M$. This extension will be presented in a forthcoming article.

In Section 2, starting from the stationary solution of the Fokker–Planck equation, we have actually determined the form of the trace-torsion and thus of the Fokker–Planck operator itself. In applying the de Rham–Kodaira–Hodge theorem, we have automatically obtained the field equations for the electromagnetic potentials which appear in the RCW geometry and its Laplacian. Clearly, we have given above the field equations for $A_{\mathrm{cocl}}$ and $A_{\mathrm{harm}}$: the first is a coclosed one-form, so it is a Maxwell field, while the second is harmonic. Both potentials couple to the exact component of the trace due to the fact that they both appear normalized by $\psi^2$. Only the coclosed term produces a nonzero electromagnetic field and satisfies the source-full Maxwell–de Rham (MdR) equation, while the harmonic component, which has a zero field, satisfies the sourceless MdR equation if we assume $\psi$ to be strictly positive; if we relax this condition so that $\psi$ has a nonempty node set, then the harmonic component satisfies the MdR equation with a source associated with a magnetic monopole current. Remarkably, these equations have incorporated the Weitzenbock term, which expresses the coupling of the electromagnetic potential to the Ricci curvature, and plays a fundamental role in the recent theory of *SU* (2) monopoles and four-manifolds structures due to Witten and coworkers. In fact, in this theory, the existence of the solutions to Witten's monopole equations depends essentially on the sign of the metric scalar curvature (Witten, 1994).

We would like to notice further that the normalized form of the electromagnetic potentials we have found, although original in its decomposition in coclosed and harmonic components obtained through the existance of a stationary density of the diffusion, is identical to the one appeared in the literature in studying the problem of coupling of torsion to the electromagnetic field, from arguments which are framed in terms of the invalidity of the minimal coupling rule when this former coupling is assumed (Hojman *et al.*, 1979).

Furthermore, this decomposition into coclosed and harmonic terms and its relation to the breaking of detailed balance in the spin plane of a Dirac–Hestenes spinor operator field (DHSOF) plays a crucial role in the equivalence between the free Maxwell equation on a Lorentzian manifold and the Dirac–Hestenes equation for a DHSOF on a RCW manifold (Rapoport, 1997); this equivalence extends the one obtained in the case that $Q$ reduces to its exact term (Rodrigues and Vaz, 1993). This equivalence allows us to think of the electromagnetic potentials of $Q$ as associated with electromagnetic potentials defined on the spin plane of a DHSOF, and thus the diffusion of spin-0 ensembles whose differential generator $H(g, \psi, A_{\mathrm{cocl}}, A_{\mathrm{harm}})$ is described by the RCW connection whose nonexact terms produce the irreversibility of these diffusions are connected with the rotational degrees of freedom of the DHSOF. It is to be remarked that the one-form $\delta\beta_2$ appears to be a generalization of the so-called Hertz potential, which in the particular case of Minkowski spacetime plays a central role in the construction of subluminal and superluminal solutions of the free Maxwell equation (Rodrigues and Lu, 1996).

Due to the fact that the geometrical structure of spacetime is an open problem in physics, torsion has appeared in different theories. Torsion has a central bearing in the theory of classical defects in condensed matter physics (Kleinert, 1989, 1991). Poincaré gauge theories of gravitation with torsion are a subject of contemporary interest, and its importance in elementary particle theory has been discussed (Hehl *et al.*, 1995). The central role of torsion in black hole theory has been elaborated (de Sabatta and Sivaram, 1991). It is sometimes alleged—as a matter of consensus—that since at a classical level the need of a gauge theory of gravitation is not manifested, there is no clear reason for the consideration of geometries with torsion as fundamental.

It is one of the main thesis of this series of articles that this consensus might be a mistake. There is no possibility of describing quantum fluctuations and nonlinear nonequilibrium classical thermodynamics in a gauge-theoretic setting unless one introduces the trace-torsion. Furthermore, one can prove that this torsion, specifically its exact component, produces a (geometrical) source for the Einstein–Cartan energy-momentum tensor derived from the metric (Rapoport, 1995a). This will allow us to connect the heat kernel of de Witt's metric approach to quantum gravity with the heat kernel associated with the exact component of the RCW connection (Rapoport, 1995d). On the ground-state Hilbert space defined by $\psi^2 \mathrm{vol}_g$, the representation of the diffusion makes explicit the exact component of the trace-torsion; yet, when transforming the RCW Laplacian by conjugation by $\psi$, the torsion is lost and appears encoded in the relativistic quantum potential, which turns out to be equal to $\frac{1}{12}R(g)$, where $R(g)$ denotes the metric scalar curvature. Therefore the metric structure of gravitation is derived from the RCW structure, which, as we have shown in this article, is related to the most general kind of quantum diffusion on spacetime. Furthermore, the nonlocality of quantum mechanics appears to be encoded in the metric scalar curvature. Remarkably, this change of representation is also at the basis of the natural linearization of the Burgers equations of hydrodynamics for a compressible fluid with null pressure (Rapoport, 1995d).

In closing this article, we would like to situate the present theory in relation to the perspective in physics due to Einstein, de Broglie, Bohm, Vigier, Nelson, and others of developing a causal theory of quantum mechanics while maintaining a stochastic theory (Holland, 1993; Selleri, 1981). Yet in this perspective, with the sole exceptions of de Broglie (1953, 1956) and Namsrai (1985), there is no hint as to the possibility that quantum fluctuations might be related to geometrical spacetime structures. In this respect, while continuing this perspective, the present approach is original; the RCW geometries are singled out, and, as discussed briefly above, these are transformed at an operator level to Riemannian representations, shedding light on the special geometrical character of Einstein's general relativity.

Rephrasing Einstein, if "God plays dice," then the distribution of the outcomes is determined by a RCW spacetime connection. Yet we must remark that the actual determination of this RCW connection requires initial and boundary conditions for solving the field equations which determine this connection.

## APPENDIX: STOCHASTIC DIFFERENTIALS AND DIFFUSION PROCESSES ON SMOOTH MANIFOLDS.

### A1. Preliminaries

There are various authoritative treatments of the subjects we shall consider in this Appendix (Rogers and Williams, 1987; McKean, 1969; Ikeda and Watanabe, 1981; Elworthy, 1982).

Let ($\Omega$, **F**, $P$) be a probability space, i.e., ($\Omega$, **F**) is a measurable space [i.e., $\Omega$ is a topological space (the "space of events"), **F** is a $\sigma$-algebra of sets of $\Omega$] and $P$ is a $\sigma$-additive nonnegative measure on $P$ such that $P(\Omega) = 1$.

Let ($S$, $B(S)$) be a topological space with the topological $\sigma$-field $B(S)$ (i.e., the smallest $\sigma$-algebra of sets containing all open sets). Recall that a mapping $X$: $\Omega \rightarrow S$ is called **F**/$B(S)$-measurable if $f^{-1}(B) = \{x/f(x) \in B\} \in B(S)$ for all $S \in B(S)$. Then an $S$-valued random variable on ($X$, **F**, $P$) is a mapping $X$: $\Omega \rightarrow S$ which is **F**/$B(S)$-measurable. In particular, if $S = R$ or $S = R^n$, then $X$ is called a real random variable or an $n$-dimensional random variable; the case which will interest us is $S$ an $n$-dimensional smooth manifold $M$, or the set of continuous curves on $M$. If $X$ an $S$-valued random variable, then

$$P^X(B) = P[X^{-1}(B)] = P[\omega: X(\omega) \in B] = P[X \in B], \qquad B \in B(S)$$

defines a probability on ($S$, $B(S)$); $P^X$ is called the probability law (or probability distribution of the random variable $X$); it is nothing else than the "induced measure" or "image measure" of the measurable mapping $X$. Let $X$, $Y$ be real random variables on a probability space ($\Omega$, **F**, $P$). Two random variables $X$ and $Y$ are identified if $P[\omega: X(\omega) \neq Y(\omega)] = 0$; a random variable $X$ is called integrable if

$$\int_\Omega |X(\omega)|\, P(d\omega) < \infty$$

For an integrable random variable, the *expectation* or *mean value* of $X$ is $E(X) := \int_\Omega X(\omega)\, P(d\omega)$. For a square-integrable random variable $X$ [i.e., $\int_\Omega |X(\omega)|^2\, P(d\omega) < \infty$], we define the *variance* of $X$:

$$V(X) = E(X^2) - E(X)^2 \quad [= E(X - E(X))^2]$$

Let $W^d = C([0, \infty) \to R^d)$ be the set of all continuous functions $w$: $[0, \infty) \ni \tau \to w(\tau) \in R^d$. One can introduce a natural metric on $W^d$ so that it turns to be a complete separable metric space under this metric. Let $B(W^d)$ be the topological $\sigma$-algebra. By a *Borel cylinder set* we mean a set $B \subset W^d$ of the form

$$B = \{W \in W^d: (w(\tau_1), \ldots, w(\tau_d)) \in E\}$$

for some sequence $0 \leq \tau_1 \leq \ldots \leq \tau_d$ and $E \in B(R^{nd})$. The mapping $w \in W^d \mapsto (w(\tau_1), \ldots, w(r_d)) \in R^{nd}$ is continuous. The totality of Borel sets coincides with $B(W^d)$.

By a $d$-dimensional continuous process $X$, we mean a $W^d$-random variable defined on a probability space $(\Omega, \mathbf{F}, P)$, i.e., a mapping $X: \Omega \to W^d$ which is $\mathbf{F}/B(W^d)$-measurable. The value at $\tau \in [0, \infty)$ of $X(w)$ is denoted by $X_\tau(w)$ or $X(\tau, w)$. For fixed $\tau$, the mapping $\omega \mapsto X_\tau(w)$ is a $d$-dimensional random variable. Conversely, a collection $\{X_\tau(w)\}_{\tau \in [0,\infty)}$ of $d$-dimensional random variables determines a $d$-dimensional continuous process if $\tau \mapsto X_\tau(w)$ is continuous with probability 1.

Let $(\Omega, \mathbf{F}, P)$ be a probability space, and $(\mathbf{F}_\tau)_{\tau\geq 0}$ be an increasing family of sub $\sigma$-fields of $\mathbf{F}$, i.e., $\mathbf{F}_\tau \subset \mathbf{F}_s$, if $0 \leq \tau \leq s$. $(\mathbf{F}_\tau)_{\tau\geq 0}$ is called *right-continuous* if $\mathbf{F}_{\tau+0} := \bigcap_{\epsilon>0} \mathbf{F}_{\tau+\epsilon} = \mathbf{F}_\tau$, $\forall \tau \in [0, \infty)$. If this is the case, we say that $\{\mathbf{F}_\tau\}_{\tau\geq 0}$ is a *reference family.*

Let $\tilde{T} = T \cup \{\infty\}$ be the one-point compactification of $T = [0, \infty)$. Let $(\Omega, \mathbf{F}, P)$ be a probability space provided with a reference family $(\mathbf{F}_\tau)_{\tau\in\tilde{T}}$. A continuous process $X = (X_\tau)_{\tau\geq 0}$ is called *adapted* to $(\mathbf{F}_\tau)_{\tau\geq 0}$ if $X_\tau$ is $\mathbf{F}_\tau$-measurable for every $\tau$. Generally, a process $X = (X_\tau)_{\tau\geq 0}$ is called *measurable* if the map $[0, \infty) \times \Omega \to R^n$, $(\tau, w) \mapsto X_\tau(w)$, is $B([0, \infty)) \times \mathbf{F}/B(R^d)$-measurable. A process $X = (X_\tau)$ is called *predictable* with respect to $(\mathbf{F}_\tau)_{\tau\geq 0}$ if the mapping $(\tau, w) \mapsto X_\tau(w)$ is $\mathbf{S}/B(R^n)$-measurable, where $\mathbf{S}$ is the smallest $\sigma$-algebra of $[0, \infty) \times \Omega$ such that all left-continuous $(\mathbf{F}_\tau)$-adapted processes are measurable. Clearly, predictable processes are measurable and adapted to $(\mathbf{F}_\tau)_{\tau\geq 0}$.

A real stochastic process $X = (X_\tau)_{\tau\in\tilde{T}}$ is called a martingale with respect to $(\mathbf{F}_\tau)_{\tau\in\tilde{T}}$ if for every $\tau \in \tilde{T}$:

(i) $X_\tau$ is integrable
(ii) $X_\tau$ is $\mathbf{F}_\tau$-measurable
(iii) $E(X|\mathbf{F}_\tau) = X_\tau$

We now introduce the basic notion of a standard Brownian motion or a Wiener process. Let $p(\tau, x)$, $t > 0$, $x \in R^d$ be

$$p(\tau, x) = (2\tau\pi)^{-d/2} \exp[-|x|^2/2\tau]$$

and let $B = (B_\tau)_{t\geq 0}$ be a $d$-dimensional continuous process such that for every $0 < \tau < \ldots < \tau_m$ and $E_i \in B(R^d)$, $i = 1, \ldots, m$, we have

$$P[B_1 \in E_1, B_{\tau_2} \in E_2, \ldots, B_{\tau_m} \in E_m]$$

$$= \int_{\mathrm{IR}^d} \mu(dx) \int_{E_1} p(\tau_1, x_1 - x)\, dx_1 \int_{E_2} p(\tau_2 - \tau_1, x_2 - x_1)\, dx_2 \ldots$$

$$\times \int_{E_m} p(\tau_m - \tau_{m-1}, x_m - x_{m-1})\, dx_m$$

where $\mu$ is a probability measure on $(R^d, B(R^d))$. Then $B = (B_\tau)$ is a $d$-dimensional continuous process such that for every $0 = \tau_0 < \tau_1 < \ldots < \tau_m$, $B_{\tau_0}, B_{\tau_1} - B_{\tau_0}, \ldots, B_{\tau_m} - B_{\tau_{x-1}}$ are mutually independent variables,

$$P^{B_0} = \mu \quad \text{and} \quad P^{B_{\tau_i} - B_{\tau_{i-1}}} = P(\tau_i - \tau_{i-1}, x_i)\, dx_i, \quad i = 1, \ldots, m$$

Such that a process is called a standard $d$-dimensional *Brownian motion* (or *Wiener process*) with the initial distribution $\mu$. The probability law $P^B(\mu)$ on $(W^d, B(W^d))$ is called the $d$-dimensional Wiener measure with the initial distribution $\mu$. Thus, the $d$-dimensional Wiener measure $P$ with the initial law $\mu$ is characterized by the property that

$$P[w\colon w(\tau_1) \in E_1, \ldots, w(\tau_m) \in E_m]$$

$$= \int_{\mathrm{IR}^d} \mu(dx) \int_{E1} p(\tau_1, x_1 - x)\, dx_1 \ldots \int_{Em} p(\tau_n - \tau_{m-1}, x_m - x_{m-1})\, dx$$

For any probability law $\mu$ on $(R^d, B(R^d))$, the $d$-dimensional Wiener measure $P^B(\mu)$ with initial distribution $\mu$ exists uniquely.

On the probability space $(W^d, B(W^d), P(\mu))$, the *coordinate process* $X(\tau, w) = w(\tau)$, $w \in W^d$, defines a $d$-dimensional Brownian process with initial law $\mu$. This process is called the *canonical realization* of a $d$-dimensional Brownian process.

The basic fact of Brownian motion is that the function $\tau \mapsto B(\tau)$ is nowhere differentiable with probability 1, and thus the integral $\int_0^\tau f(s)\, dB(s)$, for $f$ a real-valued function, is not defined in the usual sense. Thus arises the necessity of developing an extended integration theory for Brownian motions, or, more generally, for arbitrary martingales, which we briefly introduce next.

## A2. Stochastic Differentials and the Itô and Stratonovich Stochastic Integrals

Let $(\Omega, \mathbf{F}, P)$ be a probability space with right continuous increasing family $(\mathbf{F}_\tau)_{\tau\geq 0}$ of sub-$\sigma$-algebras of $\mathbf{F}$, each containing all $P$-null sets (i.e., those sets with zero $P$-measure). Let $L_2$ be the space of all real, measurable processes $\Phi = (\Phi(\tau, \omega))_{\tau \geq 0}$ on $\Omega$ adapted to $(\mathbf{F}_\tau)$ such that for every $T > 0$

$$\|\Phi\|_{2,T}^2 = E\left[\int_0^T \Phi^2(s, \omega)\, ds\right] < \infty$$

We identify $\Phi$ and $\Phi'$ in $L_2$ if $\|\Phi - \Phi'\|_{2,T} = 0\ \forall T > 0$, and then write $\Phi = \Phi'$.

Let $L_0$ be the subcollection of all real processes $\Phi = (\Phi(\tau, \omega)) \in L_2$ with the property that there exists a sequence of real numbers $0 = \tau_0 < \tau_1 < \ldots < \tau_n < \ldots \to \infty$ and a sequence of random variables $\{f_i(\omega)\}$ such that $f_i$ is $\mathbf{F}_{\tau_i}$-measurable, $\|\sup f_i\| < \infty$, and

$$\Phi(\tau, \omega) = \begin{cases} f_0(\omega) & \text{if} \quad \tau = 0 \\ f_i(\omega) & \text{if} \quad \tau \in (\tau_i, \tau_{i+1}) \end{cases}$$

Clearly, such $\Phi$ is expressed as

$$\Phi(\tau, \omega) = f_0(\omega) I_{\{\tau=0\}}(\tau) + \sum_{i=1}^{\infty} f_i(\omega) I_{(\tau_i, \tau_{i+1})}(\tau)$$

where $I_A(\tau)$ is the characteristic function of the set $A$, equal to 1 for $\tau \in A$, and 0 otherwise. It can be proved that $L_0$ is dense in $L_2$ with respect to the metric $\|\cdot\|_2$.

We define the space

$$\mathbf{M}_2 = \{X = (X_\tau)_{\tau \geq 0},\ X \text{ is a square-integrable martingale on } (\Omega, \mathbf{F}, P) \text{ with respect to } (\mathbf{F}_\tau)_{\tau \geq 0} \text{ and } X_0 = 0 \text{ almost surely (a.s.)}\}$$

and

$$\mathbf{M}_2^c = \{X \in \mathbf{M}_2, \tau \mapsto X(\tau) \text{ is a continuous a.s.}\}$$

We identify two $X, X' \in \mathbf{M}_2$ if $\tau \mapsto X_\tau$ and $\tau \mapsto X'_\tau$ coincide a.s.

We introduce the Itô stochastic integral by first introducing the stochastic integral with respect to an $(\mathbf{F}_\tau)$-Brownian motion as a mapping

$$L_2 \ni \Phi \mapsto I(\Phi) \in \mathbf{M}_2^c$$

Suppose we are given an $(\mathbf{F}_\tau)$-Brownian motion $B = (B(\tau))$ on $(\Omega, \mathbf{F}, P)$. If $\Phi \in L_0$ and

$$\Phi(\tau, \omega) = f_0(\omega) I_{\{\tau=0\}}(\tau) + \sum_{i=0}^{\infty} f_i(\omega) I_{(\tau_i, \tau_{i+1})}(\tau)$$

we then set

$$I(\Phi)(\tau, \omega) = \sum_{i=0}^{n-1} f_i(\omega)(B(\tau_{i+1}, \omega) - B(\tau_i, \omega))$$

$$+f_n(\omega)(B(\tau, \omega) - B(\tau_n, \omega))$$

for $\tau_n \leq \tau \leq \tau_{n+1}$, $n \in$ IN. Clearly, $I(\Phi)$ can be expressed as

$$I(\Phi)(\tau) = \sum_{i=0}^{\infty} f_i(B(\tau \wedge \tau_{i+1}) - B(t_1 \wedge \tau))$$

This sum is, in fact, a finite sum; here $\wedge$ denotes the supremum.

The $I(\Phi) \in \mathbf{M}_2^c$ defined likewise is called the *stochastic integral* of $\Phi \in L_2$ *with respect to the Brownian motion* $B(\tau)$, and is denoted as $\int_0^\tau \Phi(s, \omega) dB(s, w)$, or as $\int_0^\tau \Phi(s) dB(s)$. Clearly,

$$I(\alpha\Phi + \beta\Psi)(\tau) = \alpha I(\Phi)(\tau) + \beta I(\Psi)(\tau)$$

for each $\tau \geq 0$, a.s., and $\Phi$, $\Psi \in L_2$, $\alpha$, $\beta \in R$. Thus, we remark that the stochastic integral $I(\Phi)$ is a stochastic process; in fact, it is a martingale; for a fixed $\tau$, the random variable $I(\Phi)(\tau)$ is also called a stochastic integral.

The first natural extension of this stochastic process comes from the fact that physical processes have a mean motion in addition to the fluctuation described by the Brownian process. (This is a recourse to 'intuition' or even to experimental observation; it is the central thesis of this article that these two features are unified through a RCW connection.) Thus, a typical example is the stochastic process of the form

$$X(\tau) = X(0) + \int_0^\tau f(s)\, ds + \int_0^\tau g(s)\, dB\,(ds)$$

where $f(s)$ and $g(s)$ are suitable adapted process, and the last term is the stochastic integral with respect to the Brownian motion $B = (B(\tau))$ defined above. Here the process $\int_0^\tau f(s)\, ds$ describes the mean motion, and thus $X(\tau)$ as above describes a process decomposable into a process of bounded variation and a martingale. This structure is generalized to define a class of stochastic processes called semimartingales.

Let $(\Omega, \mathbf{F}, P)$ be a probability space, $(\mathbf{F}_\tau)_{\tau\geq 0}$ a reference family. Denote the following:

**M** := family of all continuous locally square-integrable martingales $M = (M_\tau)$ relative to $(\mathbf{F}_\tau)$ such that $M_0 = 0$, a.s.

**A** := family of all continuous $(\mathbf{F}_\tau)$-adapted processes $A = (A_\tau)$ such that $A_0 = 0$, and $\tau \mapsto A_\tau$ is of bounded variation on every finite interval, a.s.

**B** := family of all $(\mathbf{F}_\tau)$-predictable processes $\Phi = (\Phi_\tau)$ such that, with probability 1, $\tau \mapsto \Phi_\tau$ is bounded on each bounded interval.

Let $X = (X_\tau)$ be a continuous semimartingale, i.e., a process represented in the form

$$X_\tau = X_0 + M_\tau + A_\tau \tag{A.1}$$

where $X_0$ is an $\mathbf{F}_0$-measurable random variable, $M = (M_\tau) \in \mathbf{M}$, and $A = (A_\tau) \in \mathbf{A}$. Continuous semimartingales are also called quasimartingales, and the space of all quasimartingales is denoted as $\mathbf{Q}$. Every $X \in \mathbf{Q}$ is expressed uniquely (the Doob–Meyer canonical decomposition) as in (A.1); $M = M_X$ is called the *martingale part* and $A = A_X$ is called the *bounded variation part* of $X$.

For $X, Y \in \mathbf{Q}$, we say that $X$ and $Y$ are equivalent and write $X \sim Y$ if, with probability 1,

$$X(\tau) - X(s) = Y(\tau) - Y(s)$$

Clearly, $\sim$ is an equivalence relation. The equivalence class of $X$ is denoted by $dX$, and is called the *stochastic differential* of $X$. Then, $\int_s^\tau dX(\omega)$ is, by definition, the process $X(\tau) - X(s)$.

Let $d\mathbf{Q} = \{dX: X \in \mathbf{Q}\}$, $d\mathbf{M} = \{dM, M \in \mathbf{M}\}$ and $d\mathbf{A} = \{dA, A \in \mathbf{A}\}$. We introduce the following operations on $d\mathbf{Q}$:

$$\text{A (addition)} \quad dX + dY = d(X + Y), \qquad X, Y, \in \mathbf{Q} \tag{A.2}$$

$$\text{P (product)} \quad dX \cdot dY = d < M_X, M_Y >, \qquad X, Y \in \mathbf{Q} \tag{A.3}$$

where $M_X$, $M_Y$ are the martingale parts of $X$ and $Y$, respectively, and then $\langle M_X, M_Y \rangle$ is the covariance of $X$ and $Y$. Finally, we introduce

$B$ (multiplication) if $\Psi \in B$, $X \in \mathbf{Q}$, then

$$(\Phi \cdot X) = X(0) + \int_0^\tau \Phi(s, \omega)\, dM_X(s) + \int_0^\tau \Phi(s, \omega)\, dA_X(s), \qquad \tau \geq 0 \tag{A.4}$$

is an element in $\mathbf{Q}$. Hence, $d(\Phi \cdot X)$ is uniquely defined. We now define B-multiplication: an element $\Phi \cdot dX$ of $d\mathbf{Q}$ is defined by

$$\Phi \cdot dX = d(\Phi \cdot X) \tag{A.5}$$

Now, we can prove that $d\mathbf{Q}$, with the operations of addition, product, and B-multiplication, is a commutative algebra over $\mathbf{B}$, i.e., a commutative ring with the operations A and P that satisfy the relations:

$$\Phi \cdot (dX + dY) = \Phi \cdot dX + \Phi \cdot dY$$

$$\Phi \cdot (dX \cdot dY) = (\Phi \cdot dX) \cdot dY$$

$$(\Phi + \Psi) \cdot dX = \Phi \cdot dX + \Psi \cdot dX$$

$$(\Phi\Psi) \cdot dX = \Phi \cdot (\Psi \cdot dX)$$

for $\Phi, \Psi \in \mathbf{B}$ and $dX, dY \in d\mathbf{Q}$.

Also

$$d\mathbf{Q} \cdot d\mathbf{Q} \subset d\mathbf{A}$$

$$d\mathbf{A} \cdot d\mathbf{Q} = 0$$

$$d\mathbf{Q} \cdot d\mathbf{Q} \cdot d\mathbf{Q} = 0$$

These properties follow straightforwardly: the first from the property of stochastic integrals that $d\mathbf{Q}$ is a commutative algebra over $\mathbf{B}$, and the latter two from the fact that $\langle M_X, M_Y \rangle \in \mathbf{A}$, for $X, Y \in \mathbf{Q}$.

The stochastic calculus of Itô hinges on the Itô formula, which we give next; for a proof see Gardiner (1993). Let $X_1, \ldots, X_d \in \mathbf{Q}$, $Y = f(X_1, \ldots, X_d) \in \mathbf{Q}$ with $f$: $C^2(R^d \to R)$. Then

$$dY = \sum (\partial_i f) \cdot dX^i + \frac{1}{2} (\partial_i \partial_j) \cdot dX^i \cdot dX^d \qquad \text{(A.6)}$$

where

$$\partial_i f = \frac{\partial f}{\partial x_i} (X_1, \ldots, X_d)$$

$$\partial_i \partial_j f = \frac{\partial^2 f}{\partial x_i \partial x_j} (X_1, \ldots, X_d)$$

Wiener processes can be characterized by the stochastic differentials. If $dW_1, \ldots, dW_d \in d\mathbf{M}$ and $dW_i \cdot dW_j = \delta_{ij} d\tau$, $i, j = 1, \ldots, d$, then $(W_1(\tau), \ldots, W_d(\tau))$ is a $d$-dimensional Wiener process.

We finally introduce a fourth operation, called symmetric $Q$-multiplication.

$$\mathrm{Y} \circ \mathrm{dX} = X \cdot dX + \frac{1}{2} dX \cdot dY$$

$$= Y \cdot dX + \frac{1}{2} d \langle M_X, M_Y \rangle \qquad \text{(A.7)}$$

for $X$ and $Y \in \mathbf{Q}$.

We can now prove that $d\mathbf{Q}$, with the operations of addition, multiplication, and symmetric Q-multiplication, is a commutative algebra over $\mathbf{Q}$, i.e., for $X, Y, Z \in \mathbf{Q}$, we have

$$X \circ (dY + dZ) = X \circ dY + X \circ dZ$$
$$(X + Y) \circ dZ = X \circ dZ + Y \circ dZ$$
$$X \circ (dY \cdot dZ) = (X \circ dY) \cdot dX = X \cdot (dY \cdot dZ)$$
$$(X \cdot Y) \circ dZ = X \circ (Y \circ dZ)$$

For this, note that since $d\mathbf{Q} \cdot d\mathbf{A} = 0$, we have

$$X \circ dY = X \cdot dY$$

if $X$ or $Y \in \mathbf{A}$, and

$$\begin{aligned}(Z \circ dX) \cdot dY &= (Z \cdot dX) \cdot dY + \frac{1}{2}(dZ \cdot dX) \cdot dY \\ &= (Z \cdot dX) \cdot dY\end{aligned}$$

since $d\mathbf{Q} \cdot d\mathbf{Q} \cdot d\mathbf{Q} = 0$. Then,

$$\begin{aligned}X \circ (Y \circ dZ) &= X \cdot (Y \circ dZ) + \frac{1}{2} dX \cdot d(Y \circ Z) \\ &= X \cdot \{(Y \cdot dZ) + \frac{1}{2} dY \cdot dZ\} + \frac{1}{2} dX \cdot (Y \cdot dZ) \\ &= X \cdot (Y \cdot dZ) + \frac{1}{2}\{X \cdot (dY \cdot dZ) + dX \cdot (Y \cdot dZ)\} \\ &= X \cdot (Y \cdot dZ) + \frac{1}{2} d(X \cdot Y) \cdot dZ \\ &= (X \cdot Y) \circ dZ\end{aligned}$$

The other properties follow easily.

The main reason for introducing $\circ$ is the most remarkable fact that the Itô formula now gives the usual Newton–Leibniz transformation rules. Namely, if $X_1, \ldots, X_d \in \mathbf{Q}$ and $f \in C^3(R^d \to R)$, then $Y = f(X_1, \ldots, X_d)$ satisfies

$$dY = \partial_i f \circ dX^i$$

Indeed,

$$\partial_i f \circ dX^i = \partial_i f \cdot dX^i + \frac{1}{2}\, d(\partial_i f) \cdot dX^i$$
$$= \partial_i f \cdot dX^i + \frac{1}{2}(\partial_i \partial_j f \cdot dX^j \cdot dX^i + \partial_j \partial_k \partial_i f \cdot dX^j \cdot dX^k \cdot dX^i)$$

by the Itô formula; yet, since $d\mathbf{Q} \cdot d\mathbf{Q} \cdot d\mathbf{Q} = 0$, the third term in the previous expression vanishes and from (A.6) we further obtain

$$\partial_i f \cdot dX_i + \frac{1}{2}\, \partial_i \partial_j f \cdot dX_i \cdot dX_j = dY \tag{A.8}$$

We can now introduce the Stratonovich integral as the stochastic integral $\int_0^\tau Y \circ dX$. Then, one proves that for any $X, Y \in \mathbf{Q}$,

$$\int_0^\tau Y \circ dX = \overline{\lim_{|\delta| \to 0}} \sum_{i=1}^{n} \frac{Y(\tau_i) + Y(\tau_{i-1})}{2}\, [X(\tau_i) - X(\tau_{i-1})]$$

## A3. Diffusion Processes on Manifolds

We wish to introduce now a class of stochastic processes called diffusion processes, which are characterized by two properties, namely continuity of trajectories and the Markov property (Ikeda and Watanabe, 1981; Elworthy, 1982).

Formally, they can be introduced as follows. Let $S$ be a topological space, to which we attach a terminal point $\Delta$ so that $S' = S \cup \{\Delta\}$ is called the state space; in our constructions, $S'$ will be the one-point compactification of spacetime. Let $\overline{W}(S)$ be the set of all mappings $w$: $[0, \infty) \to S'$ such that there exists $0 \leq \xi(w) \leq \infty$ with the following properties:

(i) $w(\tau) \in S \; \forall \tau \in [0, \xi(w)]$, and the mapping $\tau \in [0, \xi(w)] \mapsto w(\tau)$ is continuous

(ii) $w(\tau) \equiv \Delta \; \forall \tau \geq \xi(w)$.

$\xi(w)$ is called the lifetime of the trajectory $w$. We set here $w(\infty) = \Delta$ $\forall w \in \overline{W}(S)$.

A Borel cylinder set in $\overline{W}(S)$ is defined for some integer $n$, a sequence $0 \leq \tau_1 < \tau_2 < \ldots < \tau_n$, and a Borel subset $A$ in $S'^n = S' \times \ldots \times S'$ ($n$ times) as $\pi^{-1}_{\tau_1, \ldots, \tau_n}(A)$, where $\pi_{\tau_1, \tau_2, \ldots, \tau_n}$: $\overline{W}(S) \to S^n$, $\pi_{\tau_1, \tau_2, \ldots, \tau_n}(w) = (w(\tau_1), \ldots, w(\tau_n))$. Let $\mathbf{B}(\overline{W}(S))$ be the $\sigma$-algebra $W(S)$ generated by all Borel cylinder sets and let $\mathbf{B}_\tau(\overline{W}(S))$ be the $\sigma$-algebra generated by all cylinder sets up to time $\tau$, i.e., sets expressed in the form $\pi^{-1}_{\tau_1, \tau_2, \ldots, \tau_n}(A)$ with $\tau_n \leq \tau$. [Here we remark that for our constructions in which $S$ is spacetime, the parameter $\tau$ *is not* the relativistic spacetime coordinate, but an internal evolution order parameter, which may be thought of as a kind of a Kaluza–Klein $(n + 1)$th coordinate.] A family of probabilities $\{P_x, x \in S'\}$ on $(\overline{W}(S), \mathbf{B}(\overline{W}(S)))$ is called Markovian if:

(i) $P_x\{w: w(0) = x\} = 1 \; \forall x \in S'$.
(ii) $x \in S \mapsto P_x(A)$ is Borel-measurable for each $A \in \mathbf{B}(\overline{W}(S))$.
(iii) $\forall \tau > s \geq 0$, $A \in \mathbf{B}_s(\overline{W}(S))$, and a Borel subset $\Gamma$ in $S'$,

$$P_x(A \cap \{w: w(\tau) \in \Gamma\}) = \int_A P_{w'(s)}\{w: w(\tau - s) \in \Gamma\} \cdot P_x\,(dw')$$

for every $x \in S'$.

For a Markovian system $\{P_x, x \in S'\}$, $\tau \in [0, \infty)$, $x \in S'$, and a Borel subset $\Gamma$ of $S'$, set

$$P(\tau, x, \Gamma) = P_x\{w: w(\tau) \in \Gamma\}$$

Then, we have

$$P_x[w(\tau_1) \in A_1, w(\tau_2) \in A_2, \ldots, w(\tau_n) \in A_n]$$
$$= \int_{A_1} P(\tau_1, x, dx_1) \int_{A_2} P(\tau_2 - \tau_1, x_1, dx_2) \ldots \int_{A_n} P(\tau_n - \tau_{n-1}, x_{n-1}, dx_n)$$

for $0 < \tau_1 < \tau_2 < \ldots < \tau_n$, $A_i \in \mathbf{B}(S')$. The family $\{P(\tau, x, \Gamma)\}$ is called the transition probability of a Markovian system, and we have that two Markovian systems on $S'$ with the same transition density coincide.

Let a Markovian system $\{P_x\}$ be given. For each $\tau > 0$, we set

$$\mathbf{F}_\tau(\overline{W}(S)) = \bigcap_{\epsilon>0} \bigcap_{x\in S'} \overline{B}_{\tau+\epsilon}\mathbf{F}(\overline{W}(S))$$

$$\mathbf{F}_\infty(\overline{W}(S)) = \bigvee_{\tau>0} \mathbf{F}_\tau(\overline{W}(S))$$

A mapping $w \in \overline{W}(S) \mapsto \sigma(w) \in [0, \infty)$ is called a stopping time if for every $\tau \geq 0$, $\{w: \sigma(w) \leq \tau\} \in \mathbf{F}_\tau(\overline{W}(S))$. For a stopping time $\sigma$, set

$$\mathbf{F}_\sigma(\overline{W}(S)) = \{A \in \mathbf{F}_\infty(\overline{W}(S)): A \cap \{w: \sigma(w) \leq \tau\} \in \mathbf{F}_\tau(\overline{W}(S)) \; \forall \tau \geq 0\}$$

The Markovian system $\{P_x\}$ is called a strongly Markov system if for every $\tau \geq 0$, $\sigma$ a stopping time, $A \in \mathbf{F}_\sigma(\overline{W}(S))$, and a Borel subset $\Gamma$ of $S'$, we have that

$$P_x(A \cap \{w: w(\tau + \sigma(w)) \in \Gamma\}) = \int_A P_{w'(\sigma(w'))}[w: w(\tau) \in \Gamma]\, P_x\,(dw')$$

$\forall x \in S'$.

Then, a family of probabilities $\{P_x\}_{x\in S'}$ on $(W'(S), \mathbf{B}(\overline{W}(S)))$ is called a diffusion process if it is a strongly Markovian system.

A stochastic process $X = \{X(\tau)\}$ on $S'$ defined on a probability space $(\Omega, \mathbf{F}, P)$ is called a diffusion process on $S'$ if there exists a diffusion $\{P_x\}_{x\in S'}$ such that, for almost all $\omega$, the mapping $\tau \mapsto X(\tau)$ is in $\overline{W}(S)$ and the image measure on $\overline{W}(S)$ of this mapping coincides with $P_\mu(\cdot) = \int_{S'} P_x(\cdot,\cdot)\,\mu(dx)$, where $\mu$ is the initial distribution of $X$, i.e., the Borel measure on $S'$ defined by $\mu(dx) = P\{\omega: X(0, \omega) \in dx\}$.

If $X = X(\tau)$ is a diffusion process, we set $\xi(\omega) = \inf\{t: X(\tau) = \Delta\}$; then, with probability 1, $[0, \xi) \ni \tau \mapsto X(\tau)$ is continuous and $X(\tau) = \Delta\ \forall\tau \geq \xi$; $\xi$ is called the life of the diffusion process.

Let $C(S')$ be the Banach space of all real- or complex-valued, bounded, continuous functions defined on $S'$, and let $A$: $C(S') \to C(S')$ be a linear operator with domain $D(A)$. Let $\{P_x, x \in S'\}$ be a system of probability measures on $(\overline{W}(S), \mathbf{B}(\overline{W}(S)))$ such that $x \mapsto P_x(A)$ is Borel-measurable. Then, $\{P_x\}$ is called an $A$-diffusion if it is strongly Markov system satisfying the following conditions:

(i) $P_x\{w: w(0) = x\} = 1 \quad \forall x \in S'$.

(ii) $f(w(\tau)) - f(w(0)) - \int_0^t (Af)(w(s))\,ds$ is a $(P_x, \mathbf{B}_\tau(\overline{W}(S))$ martingale $\forall f \in D(A)$, $\forall x$.

We have the following characterization. Suppose that $\{P_x, x \in S'\}$ is a system of probability measures satisfying (i) and (ii) above. Assume further that $\{P_x\}$ is unique, i.e., for any other system of probability measures $\{P'_x\}$ on $(\overline{W}(S), \mathbf{B}(\overline{W}(S)))$ satisfying (i) and (ii) above, $P_x = P'_x$, $\forall x$. Then $\{P_x\}$ is a $A$-diffusion.

We shall be concerned with $A$-diffusions, where $A$ is a second-order differential operator

$$Af(x) = \frac{1}{2}\, g^{\alpha\beta}(x)\, \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} + B^\alpha(x)\, \frac{\partial}{\partial x^\alpha} f$$

where $g^{\alpha\beta}(x)$, $\beta^\alpha(x)$ are real, continuous functions on $R^n$, and $(g^{\alpha\beta}(x))$ is symmetric: $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, and nonnegative definite: $g^{\alpha\beta}\xi_\alpha\xi_\beta \geq 0\ \forall\xi = (\xi^\alpha) \in R^n$, and $x \in R^n$. For domain of definition of $A$ we take $C^2_c(R^n)$, the space of all twice continuously differentiable functions with compact support.

$A$-diffusions can be constructed in terms of solutions of stochastic differential equations. Let $\sigma = (\sigma^\alpha_\beta(x)) \in R^n \otimes R^n$ such that $x \mapsto \sigma(x)$ is continuous, and $g^{\alpha\beta}(x) = \sigma^\alpha_\gamma(x)\sigma^\beta_\gamma(x)$, so that $\sigma$ is a "square root" of $(g^{\alpha\beta})$. Clearly such a $\sigma$ exists, yet it may be nonunique; so we make a choice of $\sigma$ and fix it. This will *not* imply nonuniqueness of the $A$-diffusion; in fact, the diffusion $\{P_x, x \in S'\}$ turns out to be unique under adequate analytical conditions on $\sigma$ and the drift vector field $B(x)$. These analytical conditions may be very weak, such as Sobolev-like conditions, and still the stochastic flow generated by an $A$-diffusion defines diffeomorphism of $M$ (Baxendale, 1984, Carverhill and Elworthy, 1983; Rapoport, 1996d). [Strictly speaking, it is the probability law of the two-point stochastic process $\{(X_\tau(x_1), X_\tau(x_2))\}_{\tau\geq 0}$ on $S' \times S'$ which determines the $A$-diffusion; but we shall not go into this.] Consider the stochastic differential equation

$$dX^\alpha(\tau) = \sigma^\alpha_\beta(X(\tau))\, dW^\beta(\tau) + B^\alpha(X(\tau))\, d\tau, \qquad \alpha = 1, \ldots, n \quad \text{(A.9)}$$

Now, for every $x \in R^n$, there exists a solution $X(\tau)$ of (A.9) such that $X(0) = x$. By Itô's formula, $\forall f \in C^2_c(S)$

$$f(X(\tau)) - f(X)(0)) = \int_0^\tau \frac{\partial f}{\partial x^\alpha}(X(S))\sigma^\alpha_\beta(X(S))\, dW^\beta(S) + \int_0^\tau (Af)(X(S))\, ds$$

Thus, the conditions (i) and (ii) of the definition of $A$-diffusion are satisfied. The system of probabilities $\{P_x\}$ satisfying (i) and (ii) is unique, under the already cited analytical conditions on $\sigma$ and $B$. Yet, the construction we shall give below bypasses the necessity of considering a square root of the metric $g$.

Let $A_0, A_1, \ldots, A_n$ be smooth vector fields on the smooth $n$-dimensional manifold $M$. Consider the Stratanovich s.d.e.

$$dX(\tau) = A_\alpha(X(\tau)) \circ dW^\alpha(\tau) + A_0(X(\tau))\, d\tau \qquad \text{(A.10)}$$

Here $W(\tau)$ denotes a standard Wiener process. Let $\hat{M} = M \cup \{\infty\}$ be the 1-point compactification of $M$, and let $\hat{W}(M) = \{w\colon [0, \infty) \mapsto M$ such that $w(0) \in M$, and if $w(\tau) = \infty$, then $w(\tau') = \infty\ \forall \tau' \geq \tau\}$.

A solution $X = (X(\tau))$ of (A.10) is any $(\mathbf{F}_\tau)$-adapted $\hat{W}(M)$-valued process (i.e., a continuous process on $\hat{M}$ with $\infty$ as a trap) defined on a probability space provided with a reference family $(\mathbf{F}_\tau)$ and an $n$-dimensional $(\mathbf{F}_\tau)$-Wiener process $W = (W_\tau)$ with $W(0) = 0$, and such that for any compact supported smooth function on $M$, we have

$$f(X(\tau)) - f(X(0)) = \int_0^\tau (A_\alpha f)(X(S)) \circ dW(S) + \int_0^\tau (A_0 f)(X(S))\, ds$$

Note that the vector field $A_0$ is the drift.

We can now state the following fundamental result.

*Theorem A.1* Let $\{P_x\}$ be the probability law on $\hat{W}(M)$ of the solution $X = X(\tau)$ of (A.10) with the initial value $X(0) = x$. Then $\{P_x\}_{x \in M}$ is a diffusion process generated by the second-order differential operator

$$Af: = \frac{1}{2} A_\alpha(A_\alpha f) + A_0(f), \qquad \forall f \in C_0^\infty(M)$$

Let us give the proof. Indeed, from (A.10) we get

$$df(X(\tau)) = A_\alpha f(X(\tau)) \circ dW^\alpha(\tau) + (A_0 f)(X(\tau))\, d\tau \tag{A.11}$$

or, by definition of $\circ$, (A.11) becomes

$$(A_\alpha f)(X(\tau)) \cdot dw^\alpha(\tau) + \frac{1}{2} d(A_\alpha f)(X(\tau)) \cdot dw^\alpha(\tau) + (A_0 f)(X(\tau))\, d\tau \tag{A.12}$$

Now,

$$d(A_\beta f)(X(\tau)) = A_\alpha(A_\beta f)(X(\tau)) \circ dw^\alpha(\tau) + (A_0 A_\beta f)(X(t))\, d\tau$$

so that the second term in (A.12) is

$$\begin{aligned} d(A_\beta f)(X(\tau)) \cdot dw^\beta(\tau) &= A_\alpha(A_\beta f)(X(\tau)) \circ dw^\alpha(\tau) \cdot dw^\beta(\tau) + (A_0 A_\beta f)(X(\tau))\, d\tau \cdot d\omega^\beta(\tau) \\ &= A_\alpha(A_\beta f)(X(\tau)) \cdot dw^\alpha(\tau) \cdot dw^\beta(\tau) \\ &\quad + \frac{1}{2} d(A_\alpha A_\beta f)(X(\tau)) \cdot dw^\alpha(\tau) \cdot dw^\beta(\tau) \\ &\quad + (A_0 A_\beta f)(X(\tau))\, d\tau \cdot dw^\beta(\tau) \end{aligned} \tag{A.13}$$

Now, the second and third terms of (A.13) vanish, since $d\mathbf{Q} \cdot d\mathbf{Q} \cdot d\mathbf{Q} = 0$ and $d\mathbf{A} \cdot d\mathbf{Q} = 0$, respectively, and since $dw^\alpha(\tau) \cdot dw^\beta(\tau) = d(\delta^{\alpha\beta}\tau) = \delta^{\alpha\beta}$ $d\tau$, we are left in (A.13) with

$$d(A_\beta f)(X(\tau)) \cdot dw^\beta(\tau) = A_\alpha(A_\beta f)(X(\tau)) \cdot \delta^{\alpha\beta} d\tau = A_\alpha(A_\alpha f)(X(\tau))\, d\tau$$

Therefore, from (A.11)–(A.13) we conclude that

$$df(X(\tau)) = (A_\alpha f)(X(\tau)) \cdot dw^\alpha(\tau) + \frac{1}{2}(A_\alpha A_\alpha f)(X(\tau))\, d\tau + (A_0 f)(X(\tau))\, d\tau \tag{A.14}$$

so that

$$df(X(\tau)) - Af(X(\tau))\, d\tau = df(X(\tau)) - \left(\frac{1}{2} A_\alpha A_\alpha + A_0\right) f(X(t))\, d\tau$$

coincides with the martingale $(A_\alpha f)(X(\tau)) \cdot dw^\alpha(\tau)$, and thus $A$ is the infinitesimal generator of $X(\tau)$.

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